

TWENTY DIGITS OF SOME INTEGRALS OF THE PRIME ZETA FUNCTION

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ABSTRACT. The double sum $\sum_{s \geq 1} \sum_p 1/(p^s \log p^s) = 2.00666645 \dots$ over the inverse of the product of prime powers p^s and their logarithms, is computed to 24 decimal digits. The sum covers all primes p and all integer exponents $s \geq 1$. The calculational strategy is adopted from Cohen's work which basically looks at the fraction as the underivative of the Prime Zeta Function, and then evaluates the integral by numerical methods.

1. OVERVIEW

Definition 1. *The constant in the focus of this work is the sum*

$$(1) \quad C \equiv \sum_{s=1}^{\infty} \sum_p \frac{1}{p^s \log p^s}$$

over all primitive prime powers p^s of the prime numbers p [15, 2.27.3][14, §2.2].

Remark 1. *The prime powers are represented by Sloane's sequence A000961—we drop the leading term $p^s = 1$ to avoid division by zero—. The constant C and the contribution of $s = 1$ are entry A137250 and A137245 [26].*

The only aim of the work is to improve on the previous estimates $2.00 < C < 2.01$ [12] and $C > 2.006$ [4].

The simple computational strategy is to accumulate the partial sums over s in

$$(2) \quad C \equiv \sum_{s=1}^{\infty} \frac{1}{s} \sum_p \frac{1}{p^s \log p},$$

which converge at reasonable speed to an accuracy of 10^{-7} in C after 20 terms or to an accuracy of 10^{-19} after 59 terms, for example.

2. COHEN'S INTEGRAL

The implementation is a shameless replica of Henri Cohen's reduction of the double sum over exponents and primes to a series of integrals [8].

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2.1. Logarithm-to-Integral Conversion. The logarithm of Euler's formula for Riemann's zeta function [3][18, 9.523.1]

$$(3) \quad \zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

is

$$(4) \quad \log \zeta(s) = - \sum_p \log(1 - p^{-s}),$$

which turns with the Taylor expansion of the logarithm into [18, 9.523.2][21]

$$(5) \quad \log \zeta(s) = \sum_{k=1}^{\infty} \sum_p \frac{1}{k p^{ks}}.$$

Definition 2. *The Prime Zeta Function is [16, 23]*

$$(6) \quad P(s) = \sum_p \frac{1}{p^s}; \quad \Re s > 1.$$

So the penultimate equation can be rephrased as

$$(7) \quad \log \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k} P(ks).$$

The Möbius inversion of this formula reads [2, (9b)][10, §17.1.3][25]

$$(8) \quad P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks).$$

First integrals of terms in (6) are [9, 565.1.]

$$(9) \quad \int_s^{\infty} \frac{1}{p^t} dt = \frac{1}{p^s \log p}, \quad s > 1,$$

so integration of (8) using (6) gives

$$(10) \quad \int_s^{\infty} \sum_p \frac{dt}{p^t} = \sum_p \frac{1}{p^s \log p} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_s^{\infty} \log \zeta(kt) dt = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} \int_{ks}^{\infty} \log \zeta(t) dt.$$

Definition 3. *Cohen's integral is*

$$(11) \quad I(m) \equiv \int_m^{\infty} \log \zeta(t) dt$$

over the logarithm of Riemann's zeta function with variable integer lower limit [8].

Insertion into (10) outlines the strategy to evaluate (2),

$$(12) \quad \sum_p \frac{1}{p^s \log p} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} I(ks).$$

2.2. Transition to the Dirichlet Eta Function. For $I(1)$, the pole of $\zeta(t)$ at $t = 1$ can be isolated by handling the singularity via the smooth function [1, 23.2.19][27, p 267]

$$(13) \quad \eta(s) \equiv (1 - 2^{1-s})\zeta(s).$$

For lower limits of the integral larger than one, this recipe is not needed; we stick to it to present a shorter, simpler program.

Remark 2. The finite value at $s = 1$ is [15][18, 0.232.1,1.511]

$$(14) \quad \eta(1) = \log 2.$$

This follows also from the residuum of $\text{Res}\zeta(s)|_{s=1} = 1$ in conjunction with the Taylor expansion $1 - 2^{1-s} = -\sum_{l=1}^{\infty} \frac{(-\log 2)^l}{l!} (s-1)^l = (s-1) \log 2 - \frac{\log^2 2}{2} (s-1)^2 + O((s-1)^3)$. If we introduce Stieltjes constants γ_j [5, 6, 7],

$$(15) \quad \zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \frac{\gamma_2}{2}(s-1)^2 + \dots,$$

the Taylor series of η near $s = 1$ becomes

$$(16) \quad \eta(s) = \log 2 + \log 2 \left(\gamma - \frac{\log 2}{2} \right) (s-1) - \log 2 \left(\gamma_1 + \gamma \frac{\log 2}{2} - \frac{\log^2 2}{6} \right) (s-1)^2 + \dots$$

The integrated logarithm of (13) is

$$(17) \quad \int_m^{\infty} ds \log \eta(s) = \int_m^{\infty} ds \log(1 - 2^{1-s}) + \int_m^{\infty} ds \log \zeta(s).$$

One term is a dilogarithm [17, 19, 20],

$$(18) \quad \int_m^{\infty} \log(1 - 2^{1-s}) ds = \int_{1-2^{1-m}}^1 \frac{\log x}{\log 2(1-x)} dx = -\frac{1}{\log 2} \text{Li}_2\left(\frac{1}{2^{m-1}}\right).$$

Remark 3. Special values at $m = 1$ and $m = 2$ are $\text{Li}_2(1) = \pi^2/6$ and $\text{Li}_2(1/2) = \pi^2/12 - \frac{1}{2} \log^2 2$; see the constants A013661 and A076788 in the On-Line Encyclopedia of Integer Sequences [26]. An accurate representation of $\pi^2/(12 \log 2)$ is entry A100199. $\eta(1) = \log 2$ is A002162, γ is A001620, $\eta'(1) = \log 2(\gamma - \log 2/2)$ is A091812, and $-\gamma_1$ is A082633.

This moulds (17) into

$$(19) \quad I(m) = \int_m^{\infty} ds \log \eta(s) + \frac{1}{\log 2} \text{Li}_2\left(\frac{1}{2^{m-1}}\right).$$

2.3. Numerical Implementation. The variable substitution

$$(20) \quad u = 1 - \frac{m}{s}; \quad s = \frac{m}{1-u}; \quad ds = \frac{m}{(1-u)^2} du$$

maps the interval $m \leq s < \infty$ onto the interval $0 \leq u \leq 1$.

Remark 4. Alternative substitutions like $u = 1/m - m/s^2$ or $u = 1 - 2^{-s}$ also map the half-infinite s -interval to a finite u -interval. They have the advantage of more evenly balanced integral kernels, but the disadvantage of infinite slopes at one end of the u -interval.

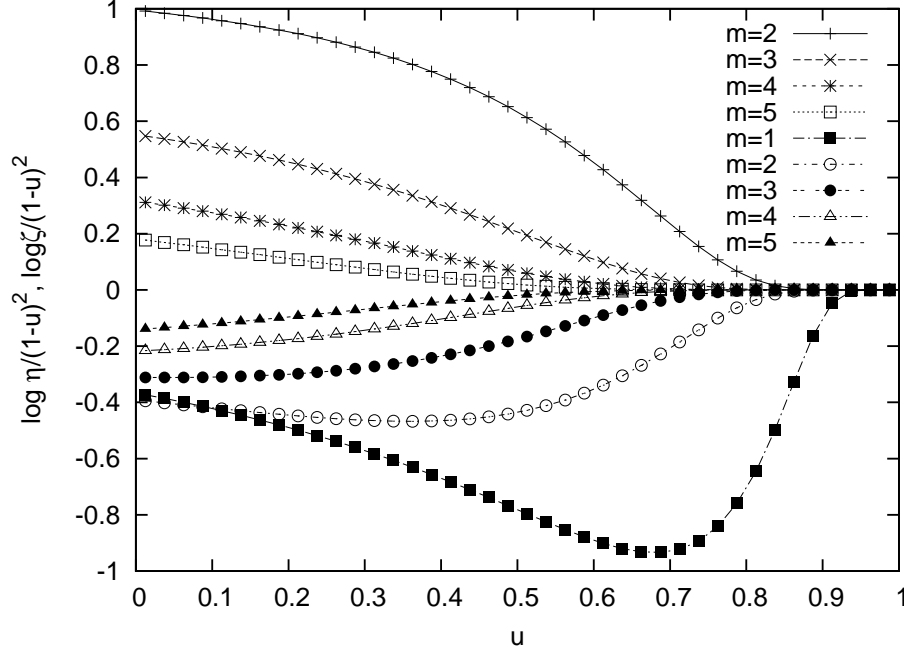


FIGURE 1. The functions $(m \log \eta)/(1-u)^2 < 0$ or $(m \log \zeta)/(1-u)^2 > 0$ arising in the integral kernels through the substitution (20).

Wynn's ϵ -algorithm [28, 22] is applied to numerical values gathered by the trapezoidal rule to evaluate the η -integral in (19),

$$(21) \quad \int_m^\infty ds \log \eta(s) = m \int_0^1 \frac{\log \eta\left(\frac{m}{1-u}\right)}{(1-u)^2} du.$$

Remark 5. Because $\log \zeta(s) \approx 2^{-s}$ at $s > 10$, $I(m) \approx 1/(2^m \log 2)$ as $m \rightarrow \infty$. This functional dependence leads to an almost straight line on the semi-logarithmic plot in Fig. 2. So the first neglected term in (2) is a good estimator to the error in the partial sums.

2.4. Intermediate Results. The following table shows s and $I(s)$ in the left double column, s and $\sum_p 1/(p^s \log p)$ in the right double column, last digits rounded:

1 1.797569958628739407930251e+00	1 1.636616323351260868569658e+00
2 5.365269459211771009617190e-01	2 5.077821878591993187743751e-01
3 2.275044530363196437654595e-01	3 2.212033403969814969599433e-01
4 1.041502253168599126451383e-01	4 1.026654782331571962664758e-01
5 4.942465209576037094607771e-02	5 4.906357474830439568004180e-02
6 2.392429939526549813915300e-02	6 2.383519825459824737569112e-02
7 1.171800374394507398697301e-02	7 1.169586557789515115400634e-02
8 5.781462903507255601662420e-03	8 5.775944595316604352784204e-03
9 2.865717415930253367688719e-03	9 2.864339771659922115926198e-03
10 1.424706500149304604154559e-03	10 1.424362320209387639441021e-03
11 7.096782687818267364308355e-04	11 7.095922515383712411613441e-04

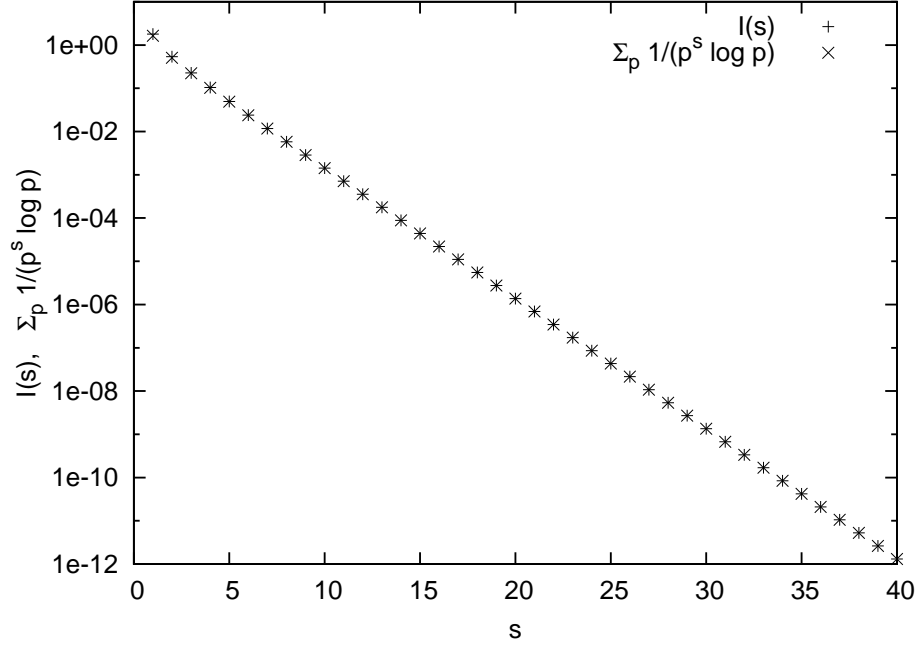


FIGURE 2. The functions $I(s)$ and $\sum_p \frac{1}{p^s \log p}$ fall off exponentially as a function of their argument s , and are not distinguishable in this common plot.

12	3.539573278954885950994965e-04	12	3.539358269254206683875757e-04
13	1.766870488681893665136623e-04	13	1.766816740292220402778692e-04
14	8.824687187128450251937842e-05	14	8.824552821042831521626792e-05
15	4.409135113515804839684079e-05	15	4.409101522588177584140913e-05
16	2.203501275340515891846297e-05	16	2.203492877680910955374197e-05
17	1.101395992552518507579192e-05	17	1.101393893146441713313451e-05
18	5.505799725855358158700238e-06	18	5.505794477350959773416061e-06
19	2.752506920732716039062161e-06	19	2.752505608607939317868454e-06
20	1.376122595515799546777009e-06	20	1.376122267484767801052807e-06
21	6.880177047831559600925881e-07	21	6.880176227754180030145763e-07
22	3.439943284950306802000172e-07	22	3.439943079930986522426868e-07
23	1.719923247093458602614288e-07	23	1.719923195838631569242761e-07
24	8.599454961284003932811501e-08	24	8.599454833146940100459182e-08
25	4.299673733467609442174442e-08	25	4.299673701433343948018079e-08
26	2.149818953720244919068455e-08	26	2.149818945711678602970514e-08
27	1.074903506531325106214723e-08	27	1.074903504529183534309235e-08
28	5.374497633245768258586266e-09	28	5.374497628240414337653354e-09
29	2.687242183906380835200363e-09	29	2.687242182655042356063437e-09
30	1.343618881152634756065114e-09	30	1.343618880839800136417081e-09
31	6.718087036690018174171263e-10	31	6.718087035907931625220498e-10
32	3.359041062052731307704559e-10	32	3.359041061857209670487929e-10
33	1.679519712278619904984955e-10	33	1.679519712229739495683419e-10
34	8.397595832274787450826827e-11	34	8.397595832152586427576250e-11
35	4.198797006441494468678066e-11	35	4.198797006410944212865828e-11

36	2.099398199991330939924248e-11	36	2.099398199983693375971239e-11
37	1.049698998919830689452100e-11	37	1.049698998917921298463855e-11
38	5.248494657681297045976310e-12	38	5.248494657676523568505703e-12
39	2.624247216535094520600355e-12	39	2.624247216533901151232704e-12
40	1.312123570832462108601650e-12	40	1.312123570832163766259737e-12
41	6.560617729378942125931467e-13	41	6.560617729378196270076685e-13
42	3.280308823095077105921359e-13	42	3.280308823094890641957663e-13
43	1.640154397682756111408190e-13	43	1.640154397682709495417266e-13
44	8.200771942197877942789073e-14	44	8.200771942197761402811764e-14
45	4.100385955693647813828150e-14	45	4.100385955693618678833822e-14
46	2.050192972711729282745366e-14	46	2.050192972711721998996784e-14
47	1.025096484644167040382940e-14	47	1.025096484644165219445795e-14
48	5.125482417515178049571403e-15	48	5.125482417515173497228539e-15
49	2.562741206855703686734819e-15	49	2.562741206855702548649103e-15
50	1.281370602793890158864587e-15	50	1.281370602793889874343158e-15
51	6.406853011856245416428711e-16	51	6.406853011856244705125138e-16
52	3.203426505223720974861043e-16	52	3.203426505223720797035150e-16
53	1.601713252377059924465468e-16	53	1.601713252377059880008995e-16
54	8.008566261102631116158675e-17	54	8.008566261102631005017492e-17
55	4.004283130290426065285105e-17	55	4.004283130290426037499809e-17
56	2.002141565058249870693295e-17	56	2.002141565058249863746971e-17
57	1.001070782500137215275764e-17	57	1.001070782500137213539183e-17
58	5.005353912404060344256377e-18	58	5.005353912404060339914924e-18
59	2.502676956169821595115832e-18	59	2.502676956169821594030469e-18
60	1.251338478074174605310912e-18	60	1.251338478074174605039571e-18
61	6.256692390335085719290662e-19	61	6.256692390335085718612310e-19
62	3.128346195155613757280562e-19	62	3.128346195155613757110974e-19
63	1.564173097573830511199490e-19	63	1.564173097573830511157093e-19
64	7.820865487855897997896814e-20	64	7.820865487855897997790822e-20
65	3.910432743923530812923694e-20	65	3.910432743923530812897196e-20
66	1.955216371960292677789151e-20	66	1.955216371960292677782526e-20
67	9.776081859796554293375620e-21	67	9.776081859796554293359059e-21
68	4.888040929896640781499146e-21	68	4.888040929896640781495005e-21
69	2.444020464947774935687030e-21	69	2.444020464947774935685995e-21
70	1.222010232473705649489420e-21	70	1.222010232473705649489161e-21
71	6.110051162367922186267000e-22	71	6.110051162367922186266353e-22
72	3.055025581183759072740187e-22	72	3.055025581183759072740025e-22
73	1.527512790591812196239003e-22	73	1.527512790591812196238962e-22
74	7.637563952958836514091412e-23	74	7.637563952958836514091310e-23
75	3.818781976479343434677847e-23	75	3.818781976479343434677822e-23
76	1.909390988239646776549639e-23	76	1.909390988239646776549633e-23
77	9.546954941198150746783920e-24	77	9.546954941198150746783904e-24
78	4.773477470599047661403870e-24	78	4.773477470599047661403866e-24
79	2.386738735299514593372572e-24	79	2.386738735299514593372571e-24
80	1.193369367649754217576498e-24	80	1.193369367649754217576498e-24

At large s , the two values equalize because the first term at $k = 1$ dominates the sum at the right hand side of (12). Cohen reported the first two rows [8]. Erdős and Zhang published an upper bound of $\sum_p 1/(p \log p)$ [11]; a proof of convergence had been given earlier [13].

The rightmost column $P^L(s, 1)$ is an aid to computation of the more general

$$(22) \quad P^L(s, a) \equiv \sum_p \frac{1}{(a-1+p)^s \log p} = \sum_{l=0}^{\infty} \binom{s+l-1}{s-1} (1-a)^l P^L(l+s, 1).$$

The l -series converges for $a = 0$ and $a = 2$. Larger a are then reached recursively with the alternating geometric series

$$(23) \quad P^L(s, a+1) = \sum_p \frac{1}{(a-1+p)^s \left(1 + \frac{1}{a-1+p}\right)^s \log p} = \sum_{l=0}^{\infty} \binom{s+l-1}{s-1} (-1)^l P^L(s+l, a).$$

This yields the following short table with double columns of s, a and $P^L(s, a)$ each:

1	0	2.564343220686309193e+00	4	2	2.200937333087681441e-02
2	0	1.735535173734295407e+00	5	2	6.924273989916642133e-03
3	0	1.569430040669505312e+00	6	2	2.216719348246691407e-03
4	0	1.502480490856796843e+00	7	2	7.177059901294852453e-04
5	0	1.471819234100400691e+00	8	2	2.341804937450834077e-04
6	0	1.457080819283874067e+00	9	2	7.683437427098209717e-05
7	0	1.449846098516752367e+00	10	2	2.531100154045370884e-05
8	0	1.446260454845751749e+00	11	2	8.362846790350413196e-06
9	0	1.444475273576531254e+00	12	2	2.769232699593665178e-06
10	0	1.443584547482253809e+00	13	2	9.185072149314254133e-07
11	0	1.443139643195342866e+00	14	2	3.050306451883795961e-07
12	0	1.442917304533643634e+00	15	2	1.013929603798279887e-07
13	0	1.442806163373811902e+00	16	2	3.372678630308857437e-08
14	0	1.442750599803600762e+00	17	2	1.122456404716334707e-08
15	0	1.442722819765431096e+00	18	2	3.737099942191877583e-09
16	0	1.442708930182166938e+00	19	2	1.244595144792613520e-09
17	0	1.442701985499337981e+00	20	2	4.145889249942980892e-10
18	0	1.442698513185098959e+00	1	3	1.083884561790478549e+00
19	0	1.442696777034769093e+00	2	3	1.539997501094314544e-01
20	0	1.442695908961300868e+00	3	3	3.279427700685188571e-02
1	2	1.280433764453964036e+00	4	3	7.457583410785408360e-03
2	2	2.518771844123868571e-01	5	3	1.747751882461897344e-03
3	2	7.208846839108258174e-02			

In the left column of this table we observe $\lim_{s \rightarrow \infty} P^L(s, 0) = 1/\log(2) \approx 1.44 \dots$

3. SUMMARY

We obtained the constant

$$(24) \quad C = 2.00666645283106875643229 \dots$$

by a rather basic numerical approach to Cohen's integral over the logarithm of the Prime Zeta Function. A table of $\sum_p 1/(p^s \log p)$ was generated for $1 \leq s \leq 80$ and a table of $\sum_p 1/[(p \pm 1)^s \log p]$ for $1 \leq s \leq 20$.

APPENDIX A. PARI PROGRAM

The following PARI/GP program [24] implements the algorithm. `WynnEpsItr` and `WynnEps` calculate Wynn's generalized Richardson extrapolation of a sequence. `logEta` computes $\log \eta$. `ICohen` calculates (19). `Ieta` calculates the integral (21). `logSum` returns the sum (12) for a given argument s . `ErdosLConst` returns a value of (2).

```

/** Wynn's epsilon process. One step of the recurrence.
 * @param eps1 A column in the scheme of deltas.
 * @param eps2 The column to the right of eps1.
 * @return The extrapolation of the column eps1.
 */
WynnEpsItr(eps1,eps2)={
    /* eps3 is the new column to the right of eps2.
     * slen is the number of items in eps1.
     */
    local(eps3,slen) ;
    slen=length(eps1) ;
    eps3=vector(slen-2) ;
    /* one-by-one filling of the new column */
    for(i=1,slen-2,
        eps3[i]= eps1[i+1]+1/(eps2[i+1]-eps2[i]) ;
    ) ;
    /* Either we have reached the rightmost column, indicated by the
     * fact that only one entry is left in eps3, or we iterate
     * once more with information now in the two
     * rightmost columns, eps2 and eps3.
     */
    if ( length(eps3)==1,
        return( eps3[1]) ,
        return( WynnEpsItr(eps2,eps3)) ;
    ) ;
}

/** Wynn's epsilon process. Main entry.
 * @param S The vector of the current estimates with an odd number of terms.
 * @return The extrapolation of the vector terms to infinity.
 */
WynnEps(S)={
    /* Essentially build the 2nd column of the scheme from the
     * inverted first differences of the vector, and enter the iteration.
     */
    local(eps1,slen = length(S)) ;
    eps1=vector(slen-1) ;
    for(i=1,slen-1,
        eps1[i]=1/(S[i+1]-S[i]) ;
    ) ;
    return( WynnEpsItr(S,eps1)) ;
}

/** Logarithm of eta(s).
 * The eta function is the product of  $1-2^{-(1-s)}$  with the Riemann zeta
 * function  $\zeta(s)$ . The routine is meant to deal explicitly with
 * the value  $\eta(1)=\log(2)$  at the pole of  $\zeta(s)$ .
 */
logEta(s)={
    if(s==1,
        return( log(log(2)) ) ,
        return( log((1-2^(1-s))*zeta(s)) ) ;
    ) ;
}

```



```

}

/** Calculate Cohen's  $I(m) = \int_{s=m}^{\infty} \log \zeta(s) ds$ .
 * @param m The lower limit of the integral
 * @param relerr The relative error admissible by the result.
 */
ICohen(m,relerr)={
    local(li2,iz) ;

    /* Calculate a value of the  $\text{Li}_2(1/2^{m-1})$  into li2.
    */
    if(m==1,
        li2=Pi^2/6 ,
        if(m==2,
            li2=Pi^2/12-(log(2))^2/2 ,
            li2=dilog(1/2^(m-1)) ;
        ) ;
    ) ;

    /* Delegate the main calculation to the integral of the
    * eta-function.
    */
    iz = Ieta(m,relerr) + li2/log(2) ;
    printp("I ",m," ",iz) ;
    return ( iz ) ;
}

/** Calculate the integral  $\int_{s=m}^{\infty} \log \eta(s) ds$ .
 * Implemented as  $m \cdot \int_0^1 \log \eta(m/(1-u)) du / (1-u)^2$ .
 * @param m The lower limit of the integral
 * @param relerr Admissible relative error of the result
 * @return The value.
 */
Ieta(m,relerr)={
    /* simp0 is the value of the kernel at the lower limit  $u=0$ . u are specific
    * abscissa points. WynnL is the column number in Wynn's extrapolation, an odd
    * number equal to or larger than 1.
    */
    local(simp0=logEta(m),u,sus,N,otrap,WynnL=9,sim) ;

    /* sus[1,.] contains sums of the integral kernel of the previous
    * step size; sus[2,.] accumulates the sums at the doubled abscissa count N.
    * We keep track of the sums in two batches, sus[,1] and sus[,2] for
    * even and odd indexed abscissa points, because subdivision of the
    * old intervals in factors of 2 can use the old sums again.
    */
    sus=matrix(2,2) ;

    /* sim[1] contains the previous extrapolation to zero step size,
    * sim[2] the newer extrapolation estimated with N abscissa values.
    */
    sim=vector(2) ;

```

```

/* otrap: A history of evaluations of the integral with trapezoidal rules
* (which ensures that Wynn's extrapolation is applicable because the error is linear
* in the step width). otrap[i+1] obtained at half the step width of otrap[i].
*/
otrap=vector(WynnL) ;

N=4 ;
while(1,
  /* Accumulate sum over function values at new points (interstitial
  * to previous points) in su[2,2].
  */
  sus[2,2]=0 ;
  forstep(i=1,N-1,2,
    u = i/N ;
    sus[2,2] += logEta (m/(1-u)) /(1-u)^2 ;
  ) ;

  /* Sum of function values over the other half of the abscissa points is
  * sum of values over all points of the previous coarser grid.
  */
  sus[2,1]=sus[1,1]+sus[1,2] ;
  /* If this is the first loop, su[2,1] is zero (from the missing
  * previous loop) and is computed explicitly.
  */
  if(sus[2,1]==0,
    forstep(i=2,N-2,2,
      u = i/N ;
      sus[2,1] += logEta(m/(1-u)) / (1-u)^2 ;
    ) ;
  ) ;

  /* Shift the history stack of old values with coarser abscissa by one index */
  for(h=2,WynnL,
    otrap[h-1]=otrap[h]
  ) ;

  /* The integral is (trapez. rule) sum of function values, half weight
  * to values at both ends of the interval. Factor m from the variable substitution.
  * Factor 1/N represents the subinterval width.
  */
  otrap[WynnL] = m*(simp0/2+sus[2,2]+sus[2,1])/N ;
  if( otrap[1]!=0.,
    /* If history has reached sufficient depth defined by WynnL, compute
    * Wynn's extrapolation in sim[2], and leave loop if converged
    */
    sim[2] = WynnEps(otrap) ;

    if( abs(sim[2]-sim[1]) < relerr*abs(sim[2]),
      return(sim[2]) ;
    ) ;
  ) ;

  /* Save sums over function values of both submeshes for next iteration. */

```

```

    sus[1,1]=sus[2,1] ;
    sus[1,2]=sus[2,2] ;
    sim[1]=sim[2] ;

    /* Double number of subintervals. Half abscissa mesh width. */
    N *= 2 ;
  ) ;
}

/** Compute sum_p 1/(p^s*log(p)) over the primes p.
 * @param s The exponent in the denominator
 * @param relerr The relative error of the result
 * @return The sum including all terms until the first one
 *         skipped is smaller than the error bar.
 */
logSum(s,relerr)={
  /* a and aold are the new and previous partial sum.
   * k is the summation index. thisI is the integral I(k*s)
   */
  local(a=0,aold=0,k=1,thisI) ;
  while(1,
    /* To avoid premature termination of the summation
     * thru a zero value of the Moebius function, add a
     * check on the function in advance.
     */
    if( moebius(k) != 0,
      /* The value of I(k*s) is either in the hash
       * vector, or calculated and saved into the vector.
       */
      if( k*s > length(Ihash),
        thisI=ICohen(k*s,relerr) ,

        if( Ihash[k*s]==0,
          Ihash[k*s]=ICohen(k*s,relerr)
        ) ;
        thisI=Ihash[k*s] ;
      ) ;

      /* Update partial sum by adding mu(k)*I(k*s)/k^2
       * to the previous value.
       */
      a += moebius(k)*thisI/k^2 ;
      if( abs(a-aold) < relerr*abs(a),
        printp("log ",s," ",a) ;
        return(a) ;
      ) ;
    ) ;
    k++ ;
    aold=a ;
  ) ;
}

```

```

/** Calculate the main constant.
 * @param relerr The relative error admitted to the result.
 * Sums the individual terms of  $\sum_{s=1,2,3,\dots} 1/s \sum_{p=2,3,5,7,11,\dots} 1/(p^s \log p)$ 
 */
ErdosLConst(relerr)={
    local(a=0,aold=0,s=1) ;

    /* A specific I() integral occurs multiple times as various values
     * of the summation variable s, so gathering the values computed in
     * a hash vector saves time. Ihash[s] stores I(s) with capacity for s=1..400.
     */
    Ihash=vector(400) ;
    while(1,
        /* we are adding terms until the last one at s=sL falls below
         * the error margin. The last one is of the order  $\log(..(sL).)/sL$ 
         * where  $\log(...)$  is roughly  $1/(2^{sL} \log 2)$ . We could allow a relative
         * error larger than the original call here, but skip this
         * aspect for simplicity, not to jeopardize the use of the
         * hashed table of I() integrals.
         */
        a += logSum(s,relerr)/s ;
        print("partial sum ", s, " ", a) ;

        /* The relative errors is roughly half the absolute error,
         * so we leave the loop if the last term is smaller than the error.
         */
        if( abs(a-aold) < relerr*abs(a),
            return(a) ;
        ) ;
        aold=a ;
        s++ ;
    ) ;
}

/** Main program. */
{
    ErdosLConst(1.e-28) ;
    quit();
}

```

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